Mini-Lecture 9.5

The Dot Product

Learning Objectives:

1. Find the Dot Product of Two Vectors (p. 606)
2. Find the Angle between Two Vectors (p. 607)
3. Determine Whether Two Vectors Are Parallel (p. 608)
4. Determine Whether Two Vectors Are Orthogonal (p. 608)
5. Decompose a Vector into Two Orthogonal Vectors (p. 609)
6. Compute Work (p. 610)

Examples:

1. If $\mathbf{v} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} + 9\mathbf{j}$, find $\mathbf{v} \bullet \mathbf{w}$.

2. If $\mathbf{u} = 4\mathbf{i} - 6\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 5\mathbf{j}$, find the angle between $\mathbf{u}$ and $\mathbf{w}$.

3. If $\mathbf{u} = -\mathbf{i} - 6\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$, determine whether $\mathbf{u}$ and $\mathbf{w}$ are parallel, orthogonal, or neither.

4. Find the vector projection of $\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$ onto $\mathbf{w} = -3\mathbf{i} - 9\mathbf{j}$. Decompose $\mathbf{v}$ into two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

9.5 mini lecture

$v = a_1 i + b_1 j$
$w = a_2 i + b_2 j$

Dot Product $v \cdot w = a_1 a_2 + b_1 b_2$

$u \cdot v = v \cdot w$ Commutative dot product

$u \cdot (v + w) = u \cdot v + u \cdot w$ distributive dot product

$v \cdot v = \|v\|^2$

$0 \cdot v = 0$

Angle between vectors

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

$\theta = 90$ orthogonal

$\theta = 0, 180$ parallel

$v \cdot w = 0$ orthogonal

$$v_1 = \frac{v \cdot w}{\|w\|^2} w$$ project v onto w

$$v_1 = \frac{v \cdot w}{\|w\|^2} w \qquad v_2 = v - v_1$$

v_1 parallel to w

v_2 orthogonal to w

mini lecture 9.5 continued

① $v = -i - j$

$w = 3i + 9j$

Dot Product

$v \cdot w = (-1)(3) + (-1)(9)$

$= \boxed{-12}$

② $u = 4i - 6j$

$w = -2i + 5j$

Find angle between

$$\cos\theta = \frac{u \cdot w}{\|u\|\|w\|} = \frac{4(-2) + (-6)(5)}{\sqrt{4^2 + (-6)^2}\sqrt{(-2)^2 + 5^2}}$$

$$\cos\theta = \frac{-38}{\sqrt{52}\sqrt{29}}$$

$$\boxed{\theta = 168.1^\circ}$$

9.5 mini lecture continued

③ $u = -i - 6j$

$w = 2i - j$

parallel
orthogonal
neither

$$\cos\theta = \frac{u \cdot w}{\|u\|\|w\|} = \frac{(-1)(2) + (-6)(-1)}{\sqrt{(-1)^2 + (-6)^2}\sqrt{(2)^2 + (-1)^2}}$$

$$\cos\theta = \frac{4}{\sqrt{37}\sqrt{5}}$$

$\theta = 72.9^\circ$

neither

9.5 mini lecture continued

Vector projection of v onto w

(4) $v = -i + 5j$

$w = -3i - 9j$

v_1 (parallel)

$$\frac{v \cdot w}{\|w\|^2} w = \frac{(-1)(-3) + (5)(-9)}{\left(\sqrt{(-3)^2 + (-9)^2}\right)^2} (-3i - 9j)$$

$$= \frac{-42}{90} (-3i - 9j)$$

$$\boxed{v_1 = \frac{7}{5} i + \frac{21}{5} j}$$ parallel

v_2 (orthogonal)

$$v - v_1 = (-i + 5j) - \left(\frac{7}{5} i + \frac{21}{5} j\right)$$

$$\boxed{v_2 = -\frac{12}{5} i + \frac{4}{5} j}$$

Check

parallel

$\alpha\left(\frac{7}{5} i + \frac{21}{5} j\right) = (-3i - 9j)$

$\alpha = \frac{-15}{7}$ multiplier ✓

orthogonal

$\left(-\frac{12}{5}\right)(-3) + \left(\frac{4}{5}\right)(-9)$

$= 0$

$v \cdot w = 0$ ✓